Pergamon

Computers Math. Applic. Vol. 29, No. 12, pp. 67–73, 1995
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0898-1221/95 \$9.50 + 0.00

0898-1221(95)00059-3

A Note on Parametric Spline Function Approximation

C. V. Raghavarao and S. T. P. T. Srinivas
Department of Mathematics
Indian Institute of Technology, Madras, 600 036, India

(Received May 1994; accepted June 1994)

Abstract—The application of a parametric spline function which depends on a parameter $p > 0$ to two linear partial differential equations is discussed. For $p = 0$, the parametric spline reduces to the ordinary cubic spline.

PARAMETRIC SPLINE FUNCTION

We consider a mesh with knots $a = x_0 < x_1 < x_2 < \cdots < x_N = b$, with $h = x_i - x_{i-1}$, $i = 1, 2, \ldots, N$. A function $S(x)$ of class $C^2[a,b]$, which interpolates $y(x)$ at the knots $\{x_i\}$, depends on a parameter $p > 0$ and reduces to a cubic spline function in the intervals $[x_{i-1}, x_i]$, as $p \to 0$, is termed a parametric spline function, as given by Jain *et al.* [1]. If $S(x)$ is a parametric spline function, then, in general,

$$S''(x) + p\,S(X) = (S''(x_{i-1}) + p\,S(x_{i-1}))\,\frac{(x_i - x)}{h} + (S''(x_i) + p\,S(x_i))\,\frac{(x - x_{i-1})}{h}, \tag{1}$$

where the prime denotes differentiation with respect to x, $S(x_i) = y(x_i)$, and p is a parameter.

Solving the differential equation (1) on $[x_{i-1}, x_i]$, we obtain

$$\begin{aligned} S(x) = &-\frac{h^2}{w^2 \sin w}\left[S''(x_i)\,\sin w\left(\frac{x - x_{i-1}}{h}\right) + S''(x_{i-1})\,\sin w\left(\frac{x_i - x}{h}\right)\right] \\ &+ \frac{h^2}{w^2}\left[\left(\frac{x - x_{i-1}}{h}\right)\left(S''(x_i) + \frac{w^2}{h^2}\,S(x_i)\right) + \left(\frac{x_i - x}{h}\right)\left(S''(x_{i-1}) + \frac{w^2}{h^2}\,S(x_{i-1})\right)\right], \end{aligned} \tag{2}$$

where $w = h\sqrt{p}$.

The constants of integration have been determined by evaluating $S(x)$ at x_{i-1} and x_i. The continuity of the first derivative of $S(x)$ at x_i gives, after simplification,

$$y_{i+1} - 2y_i + y_{i-1} = h^2\,(\alpha\,M_{i+1} + 2\beta\,M_i + \alpha\,M_{i-1}), \tag{3}$$

where $\alpha = (1/w^2)\,(w/\sin w - 1)$, $\beta = 1/w^2\,(1 - (w/\sin w)\,\cos w)$, and $M_i = S''(x_i)$.

This investigation was carried out under the C.S.I.R. project number 25/54/90-EMR-II. S.T.P.T. Srinivas is a Junior Research Fellow in the above project. The authors are grateful to C.S.I.R-EMR-II for the financial assistance.

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Additional spline relations useful in solving boundary value problems are

$$m_i = -h\,(\alpha\, M_{i+1} + \beta\, M_i) + \frac{y_{i+1} - y_i}{h}, \tag{4}$$

$$m_{i+1} = h\,(\alpha\, M_i + \beta\, M_{i+1}) + \frac{y_{i+1} - y_i}{h}, \tag{5}$$

$$m_{i+1} - m_i = (\beta + \alpha)\, h\,(M_{i+1} + M_i), \tag{6}$$

$$m_{i+1} + m_i = (\beta - \alpha)\, h\,(M_{i+1} - M_i) + \frac{2\,(y_{i+1} - y_i)}{h}, \tag{7}$$

$$\alpha\, m_{i+1} + 2\beta\, m_i + \alpha\, m_{i-1} = (\alpha + \beta)\,\frac{y_{i+1} - y_i}{h}, \tag{8}$$

where $m_i = S'(x_i)$.

Expanding the sine functions in the r.h.s. of (2) in Taylor's series for $p \to 0$, we obtain the cubic spline function

$$S(x) = \frac{(x - x_i)^3}{6h} M_{i-1} + \frac{(x - x_{i-1})^3}{6h}\, M_i + \left(y_{i-1} - \frac{h^2}{6}\, M_{i-1}\right)\left(\frac{x_i - x}{h}\right) + \left(y_i - \frac{h^2}{6}\, M_i\right)\left(\frac{x - x_{i-1}}{h}\right).$$

The parametric spline functions (2) and the corresponding relations (3)–(8) depend on a parameter w, which is to be chosen suitably. Assuming $w/2 = \tan w/2$ (the justification for this is given later), we find

$$\alpha = \beta = \frac{1}{4}.$$

The values of α and β for a cubic spline are

$$\alpha = \frac{1}{6}, \quad \beta = \frac{1}{3}.$$

Justification

Consider a second order differential equation

$$y'' = f(x, y), \qquad x \in [a, b], \tag{9}$$

with the prescribed initial conditions, $y(a)$ and $y'(a)$. Then, the multistep method corresponding to the consistency relation (3) can be used to find the approximate values of $y(x_i)$ at the nodal points x_i, $i = 1, 2, \dots, N$, where $N\, h = (b - a)$. The spline function (2) may be used to determine $y(x)$ at x_i, $i = 1, 2, \dots, N$. The multistep method equivalent to (3) is given by

$$y_{i+1} - 2y_i + y_{i-1} = \frac{h^2}{w^2}\left[\left(\frac{w}{\sin w} - 1\right) y''_{i+1} + 2\left(1 - \frac{w}{\sin w}\cos w\right) y''_i + \left(\frac{w}{\sin w} - 1\right) y''_{i+1}\right]. \tag{10}$$

It is easily verified that the functions $x, \cos\sqrt{p}\,x$, and $\sin\sqrt{p}\,x$ satisfy the relation (10) and therefore the method (10) has polynomial and trigonometric order one, for arbitrary p. Equation (10) is suitable for solving the differential equation (9), if it is consistent. This is satisfied provided that w is a root of the equation

$$\frac{w}{2} = \tan\frac{w}{2}.$$

This equation has an infinite number of roots, the smallest positive nonzero root being given by $w = 8.986818\dots$, for which the values of α and β are as given above.

PART A

Incompressible Inviscid Fluid Flow Past a Circular Cylinder (Two-Dimensional Problem)

The nondimensional differential equation governing steady, incompressible, inviscid, fluid flow past a circular cylinder, neglecting inertial terms, in polar coordinates is given by

$$\nabla^2 \psi = 0, \tag{11}$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$. Boundary conditions are

$$\begin{aligned} &\psi = 0, && \text{on } r = 1, \\ &\psi \sim r \sin\theta, && \text{as } r \to r_\infty, \\ &\psi = 0, && \text{for } \theta = 0^\circ,\ 180^\circ. \end{aligned} \tag{12}$$

Using the transformation $r = e^\xi$, Equations (11) and (12) transform to

$$\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \theta^2} = 0, \tag{13}$$

with boundary conditions:

$$\begin{aligned} &\psi = 0, && \text{for } \xi = 0, \\ &\psi \sim e^\xi \sin\theta, && \text{as } \xi \to \xi_\infty, \\ &\psi = 0, && \text{for } \theta = 0^\circ,\ 180^\circ. \end{aligned} \tag{14}$$

We take the parametric spline function approximation in ξ-direction, and finite differences (central differences) in θ-direction. We write

$$\left(\frac{\partial^2 \psi}{\partial \xi^2}\right)_{(i,j)} = M_{i,j}.$$

We next take $h = 0.1$ in ξ-direction and $k = \pi/45$ in θ-direction. ξ_∞ is taken as 3.0 such that the infinity distance is taken approximately as 20 times of the radius of the cylinder. Now, Equation (13) can be written as

$$M_{i,j} + \frac{\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}}{k^2} = 0. \tag{15}$$

The system of equations

$$M_{i,j} = -\frac{\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}}{k^2}, \tag{16}$$

$$\psi_{i,j} = \frac{1}{2}\left(\psi_{i+1,j} + \psi_{i-1,j}\right) - \frac{h^2}{2}\left(\alpha M_{i+1,j} + 2\beta M_{i,j} + \alpha M_{i-1,j}\right), \tag{17}$$

for $i = 1, 2, \dots, N-1$, $j = 1, 2, \dots, L-1$, is solved with conditions obtained from the differential equation using the boundary conditions on the body and at infinity for ψ,

$$M_{0,j} = 0, \quad M_{N,j} = -\frac{1}{k^2}\left(\psi_{N,j+1} - 2\psi_{N,j} + \psi_{N,j-1}\right), \qquad \text{and} \tag{18}$$

$$\psi_{0,j} = 0, \quad \psi_{N,j} = e^{3\xi_i} \sin\theta_j, \tag{19}$$

where $N h = 3$ and $L k = \pi$.

Table 1. Inviscid flow past a circular cylinder—Parametric Spline.

$z\backslash\theta$	16	32	48	64	80	96	112	128	144	160	176
0.2	0.1108 (0.1110)	0.2130 (0.2134)	0.2987 (0.2992)	0.3613 (0.3916)	0.3960 (0.3966)	0.3999 (0.4005)	0.3729 (0.3734)	0.3169 (0.3173)	0.2364 (0.2367)	0.1376 (0.1376)	0.0281 (0.0281)
0.4	0.2260 (0.2264)	0.4346 (0.4353)	0.6096 (0.6105)	0.7373 (0.7384)	0.8080 (0.8090)	0.8160 (0.8170)	0.7609 (0.7617)	0.6467 (0.6474)	0.4824 (0.4829)	0.2808 (0.2810)	0.0572 (0.0573)
0.6	0.3505 (0.3510)	0.6738 (0.6747)	0.9451 (0.9463)	1.1431 (1.1444)	1.2527 (1.2540)	1.2652 (1.2663)	1.1796 (1.1806)	1.0026 (1.0034)	0.7479 (0.7484)	0.4353 (0.4355)	0.0887 (0.0888)
0.8	0.4891 (0.4896)	0.9403 (0.9412)	1.3188 (1.3200)	1.5952 (1.5964)	1.7480 (1.7492)	1.7654 (1.7665)	1.6460 (1.6469)	1.3990 (1.3997)	1.0436 (1.0440)	0.6073 (0.6075)	0.1239 (0.1239)
1.0	0.6474 (0.6479)	1.2448 (1.2455)	1.7458 (1.7467)	2.1116 (2.1125)	2.3139 (2.3147)	2.3369 (2.3375)	2.1788 (2.1793)	1.8519 (1.8521)	1.3814 (1.3185)	0.8039 (0.8039)	0.1640 (0.1640)
1.4	1.0499 (1.0498)	2.0187 (2.0182)	2.8311 (2.8303)	3.4243 (3.4231)	3.7522 (3.7507)	3.7894 (3.7877)	3.5330 (3.5313)	3.0028 (3.0012)	2.2399 (2.2386)	1.3034 (1.3026)	0.2659 (0.2657)
1.8	1.6234 (1.6219)	3.1212 (3.1182)	4.3773 (4.3729)	5.2943 (5.2888)	5.8012 (5.7949)	5.8586 (5.8521)	5.4620 (5.4559)	4.6423 (4.6369)	3.4628 (3.4587)	2.0150 (2.0126)	0.4111 (0.4105)
2.2	2.4610 (2.4571)	4.7314 (4.7238)	6.6353 (6.6245)	8.0252 (8.0120)	8.7933 (8.7788)	8.8802 (8.8654)	8.2790 (8.2651)	7.0364 (7.0245)	5.2486 (5.2396)	3.0541 (3.0488)	0.6230 (0.6218)
2.6	3.6983 (3.6906)	7.1102 (7.0953)	9.9713 (9.9503)	12.0598 (12.0343)	13.2139 (13.1860)	13.3444 (13.3161)	12.4409 (12.4145)	10.5736 (10.5510)	7.8870 (7.8708)	4.5893 (4.5795)	0.9361 (0.9340)
3.0	5.5363 (5.5226)	10.6437 (10.6173)	14.9264 (14.8894)	18.0527 (18.0080)	19.7803 (19.7313)	19.9755 (19.9259)	18.6230 (18.5768)	15.8276 (15.7884)	11.8060 (11.7767)	6.8697 (6.8526)	1.4010 (1.3977)

Table 2. Inviscid flow past a circular cylinder–Cubic Spline.

$z\backslash\theta$	16	32	48	64	80	96	112	128	144	160	176
0.2	0.1109 (0.1110)	0.2132 (0.2134)	0.2990 (0.2992)	0.3616 (0.3916)	0.3963 (0.3966)	0.4002 (0.4005)	0.3732 (0.3734)	0.3172 (0.3173)	0.2366 (0.2367)	0.1377 (0.1376)	0.0281 (0.0281)
0.4	0.2262 (0.2264)	0.4350 (0.4353)	0.6101 (0.6105)	0.7379 (0.7384)	0.8086 (0.8090)	0.8167 (0.8170)	0.7615 (0.7617)	0.6472 (0.6474)	0.4828 (0.4829)	0.2810 (0.2810)	0.0573 (0.0573)
0.6	0.3507 (0.3510)	0.6744 (0.6747)	0.9458 (0.9463)	1.1440 (1.1444)	1.2536 (1.2540)	1.2661 (1.2663)	1.1805 (1.1806)	1.0034 (1.0034)	0.7485 (0.7484)	0.4356 (0.4355)	0.0888 (0.0888)
0.8	0.4894 (0.4896)	0.9410 (0.9412)	1.3198 (1.3200)	1.5963 (1.5964)	1.7493 (1.7492)	1.7667 (1.7665)	1.6472 (1.6469)	1.4001 (1.3997)	1.0444 (1.0440)	0.6078 (0.6075)	0.1240 (0.1239)
1.0	0.6479 (0.6479)	1.2456 (1.2455)	1.7470 (1.7467)	2.1131 (2.1125)	2.3155 (2.3147)	2.3385 (2.3375)	2.1803 (2.1793)	1.8532 (1.8521)	1.3824 (1.3185)	0.8044 (0.8039)	0.1641 (0.1640)
1.4	1.0506 (1.0498)	2.0198 (2.0182)	2.8328 (2.8303)	3.4263 (3.4231)	3.7544 (3.7507)	3.7916 (3.7877)	3.5350 (3.5313)	3.0046 (3.0012)	2.2412 (2.2386)	1.3042 (1.3026)	0.2660 (0.2657)
1.8	1.6242 (1.6219)	3.1227 (3.1182)	4.3793 (4.3729)	5.2967 (5.2888)	5.8038 (5.7949)	5.8612 (5.8521)	5.4645 (5.4559)	4.6444 (4.6369)	3.4644 (3.4587)	2.0159 (2.0126)	0.4112 (0.4105)
2.2	2.4617 (2.4571)	4.7329 (4.7238)	6.6374 (6.6245)	8.0277 (8.0120)	8.7961 (8.7788)	8.8830 (8.8654)	8.2816 (8.2651)	7.0386 (7.0245)	5.2502 (5.2396)	3.0550 (3.0488)	0.6231 (0.6218)
2.6	3.6989 (3.6906)	7.1114 (7.0953)	9.9729 (9.9503)	12.0617 (12.0343)	13.2161 (13.1860)	13.3465 (13.3161)	12.4429 (12.4145)	10.5753 (10.5510)	7.8882 (7.8708)	4.5900 (4.5795)	0.9362 (0.9340)
3.0	5.5363 (5.5226)	10.6437 (10.6173)	14.9264 (14.8894)	18.0527 (18.0080)	19.7803 (19.7313)	19.9755 (19.9259)	18.6230 (18.5768)	15.8276 (15.7884)	11.8060 (11.7767)	6.8697 (6.8526)	1.4011 (1.3977)

The initial values for M's and ψ's are taken as zeros. Using the SLOR–Successive Line Over Relaxation method Equation (16) with initial conditions (18) is solved for the $M_{i,j}$'s, as given in [2]. These values of $M_{i,j}$'s are used to solve Equation (17) with conditions (19), for the $\psi_{i,j}$'s using the SLOR method.

This completes one iteration. The iterations are continued till

$$\left|F^{(n+1)} - F^{(n)}\right| < 10^{-4},$$

where $F = M,\ \psi$. Table 1 gives the values obtained by the parametric spline; values in the brackets are the values of the exact solution

$$\psi = \left(r - \frac{1}{r}\right) \sin\theta.$$

Table 2 gives the values by cubic spline and, in the brackets, the exact solution values. There is good agreement in values by this method with those of the exact solution.

PART B

Stokes' Flow Past a Sphere (Axi-Symmetric Problem)

The nondimensional differential equations governing the steady, viscous, incompressible flow past a sphere, neglecting inertial terms, in spherical polar coordinates are

$$\frac{1}{r\,\sin\theta}\, D^2\psi = \zeta, \tag{20}$$

$$D^2\zeta = 0, \tag{21}$$

where $D^2 \equiv \frac{\partial^2}{\partial r^2} - \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)$.

Boundary conditions for ψ are

$$\begin{aligned} &\psi = \frac{\partial\psi}{\partial r} = 0, && \text{on } r = 1,\\ &\psi \sim \frac{1}{2} r^2 \sin^2\theta, && \text{as } r \to r_\infty,\\ &\psi = 0, && \text{for } \theta = 0^\circ,\ 180^\circ. \end{aligned} \tag{22}$$

Using the transformation $r = e^{\xi}$, Equations (20) and (21) transform to

$$e^{2\xi}\, D^2\psi - \zeta\, e^{3\xi} \sin\theta = 0, \tag{23}$$

$$e^{2\xi}\, D^2 g = 0, \tag{24}$$

where

$$e^{2\xi}\, D^2 = \frac{\partial^2}{\partial\xi^2} - \frac{\partial}{\partial\xi} + \sin\theta\,\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\right),$$

and $g = \zeta\, e^{\xi}\, \sin\theta$.

The boundary conditions for ψ are

$$\begin{aligned} &\psi = 0, && \text{for } \xi = 0,\\ &\psi = \frac{1}{2} e^{2\xi} \sin^2\theta, && \text{as } \xi \to \xi_\infty,\\ &\psi = 0, && \text{for } \theta = 0^\circ,\ 180^\circ. \end{aligned} \tag{25}$$

The boundary conditions for ζ are

$$\begin{aligned} &\zeta = 0, && \text{for } \theta = 0^\circ,\ 180^\circ,\\ &\zeta = \frac{8\psi_1 - \psi_2}{2a^2}, && \text{for } \xi = 0,\\ &\zeta = 0, && \text{as } \xi \to \xi_\infty. \end{aligned} \tag{26}$$

Table 3. Stokes' flow past a sphere—Parametric Spline.

$z\backslash\theta$	16	32	48	64	80	96	112	128	144	160	176
0.2	0.0029 (0.0026)	0.0106 (0.0097)	0.0209 (0.0191)	0.0306 (0.0279)	0.0367 (0.0335)	0.0374 (0.0342)	0.0325 (0.0297)	0.0235 (0.0214)	0.0131 (0.0119)	0.0044 (0.0040)	0.0002 (0.0002)
0.4	0.0135 (0.0123)	0.0499 (0.0453)	0.0980 (0.0892)	0.1432 (0.1305)	0.1717 (0.1566)	0.1749 (0.1597)	0.1519 (0.1388)	0.1097 (0.1003)	0.0610 (0.0558)	0.0207 (0.0189)	0.0009 (0.0008)
0.6	0.0361 (0.0327)	0.1332 (0.1209)	0.2616 (0.2378)	0.3821 (0.3479)	0.4582 (0.4177)	0.4667 (0.4260)	0.4052 (0.3702)	0.2925 (0.2674)	0.1627 (0.1488)	0.0551 (0.0504)	0.0023 (0.0021)
0.8	0.0771 (0.0699)	0.2846 (0.2583)	0.5588 (0.5079)	0.8163 (0.7430)	0.9787 (0.8920)	0.9968 (0.9096)	0.8654 (0.7906)	0.6246 (0.5711)	0.3474 (0.3177)	0.1176 (0.1076)	0.0049 (0.0045)
1.0	0.1466 (0.1328)	0.5410 (0.4908)	1.0624 (0.9652)	1.5517 (1.4119)	1.8602 (1.6951)	1.8944 (1.7287)	1.6445 (1.5025)	1.1867 (1.0853)	0.6599 (0.6038)	0.2235 (0.2045)	0.0093 (0.0085)
1.4	0.4402 (0.3983)	1.6241 (1.4722)	3.1885 (2.8953)	4.6558 (4.2351)	5.5797 (5.0845)	5.6806 (5.1853)	4.9296 (4.5069)	3.5562 (3.2554)	1.9770 (1.8113)	0.6694 (0.6133)	0.0279 (0.0255)
1.8	1.1607 (1.0487)	4.2803 (3.8761)	8.3995 (7.6230)	12.2593 (11.1506)	14.6852 (13.3870)	14.9437 (13.6524)	12.9617 (11.8662)	9.3454 (8.5713)	5.1926 (4.7689)	1.7573 (1.6147)	0.0733 (0.0672)
2.2	2.8639 (2.5820)	10.5508 (9.5433)	20.6846 (18.7683)	30.1663 (27.4536)	36.1122 (32.9596)	36.7237 (33.6130)	31.8292 (29.2153)	22.9285 (21.1030)	12.7265 (11.7413)	4.3027 (3.9754)	0.1793 (0.1654)
2.6	6.8249 (6.1204)	25.0374 (22.6215)	48.9721 (44.4887)	71.3349 (65.0763)	85.3281 (78.1278)	86.7015 (79.6767)	75.0679 (69.2523)	53.9974 (50.0227)	29.9054 (27.8317)	10.0798 (9.4234)	0.4191 (0.3920)
3.0	15.3254 (14.1819)	56.6443 (52.4175)	111.3996 (103.0871)	162.9510 (150.7918)	195.6319 (181.0340)	199.5104 (184.6231)	173.4078 (160.4683)	125.2568 (115.9103)	69.6907 (64.4904)	23.5962 (21.8354)	0.9816 (0.9083)

Table 4. Stokes' flow past a sphere—Cubic Spline.

$z\backslash\theta$	16	32	48	64	80	96	112	128	144	160	176
0.2	0.0029 (0.0026)	0.0107 (0.0097)	0.0209 (0.0191)	0.0306 (0.0279)	0.0367 (0.0335)	0.0374 (0.0342)	0.0325 (0.0297)	0.0235 (0.0214)	0.0131 (0.0119)	0.0044 (0.0040)	0.0002 (0.0002)
0.4	0.0135 (0.0123)	0.0499 (0.0453)	0.0981 (0.0892)	0.1433 (0.1305)	0.1719 (0.1566)	0.1752 (0.1597)	0.1522 (0.1388)	0.1099 (0.1003)	0.0611 (0.0558)	0.0207 (0.0189)	0.0009 (0.0008)
0.6	0.0361 (0.0327)	0.1332 (0.1209)	0.2617 (0.2378)	0.3824 (0.3479)	0.4586 (0.4177)	0.4674 (0.4260)	0.4059 (0.3702)	0.2931 (0.2674)	0.1631 (0.1488)	0.0552 (0.0504)	0.0023 (0.0021)
0.8	0.0771 (0.0699)	0.2845 (0.2583)	0.5589 (0.5079)	0.8167 (0.7430)	0.9795 (0.8920)	0.9981 (0.9096)	0.8669 (0.7906)	0.6258 (0.5711)	0.3482 (0.3177)	0.1179 (0.1076)	0.0049 (0.0045)
1.0	0.1465 (0.1328)	0.5407 (0.4908)	1.0622 (0.9652)	1.5521 (1.4119)	1.8615 (1.6951)	1.8966 (1.7287)	1.6471 (1.5025)	1.1891 (1.0853)	0.6614 (0.6038)	0.2241 (0.2045)	0.0093 (0.0085)
1.4	0.4397 (0.3983)	1.6225 (1.4722)	3.1866 (2.8953)	4.6553 (4.2351)	5.5822 (5.0845)	5.6863 (5.1853)	4.9373 (4.5069)	3.5636 (3.2554)	1.9819 (1.8113)	0.6712 (0.6133)	0.0280 (0.0255)
1.8	1.1582 (1.0487)	4.2729 (3.8761)	8.3897 (7.6230)	12.2528 (11.1506)	14.6877 (13.3870)	14.9571 (13.6524)	12.9827 (11.8662)	9.3670 (8.5713)	5.2077 (4.7689)	1.7632 (1.6147)	0.0736 (0.0672)
2.2	2.8536 (2.5820)	10.5205 (9.5433)	20.6436 (18.7683)	30.1340 (27.4536)	36.1069 (32.9596)	36.7530 (33.6130)	31.8854 (29.2153)	22.9914 (21.1030)	12.7733 (11.7413)	4.3218 (3.9754)	0.1802 (0.1654)
2.6	6.7779 (6.1204)	24.9188 (22.6215)	48.8227 (44.4887)	71.2126 (65.0763)	85.2839 (78.1278)	86.7624 (79.6767)	75.2188 (69.2523)	54.1840 (50.0227)	30.0579 (27.8317)	10.1489 (9.4234)	0.4225 (0.3920)
3.0	15.3254 (14.1819)	56.6443 (52.4175)	111.3996 (103.0871)	162.9510 (150.7918)	195.6319 (181.0340)	199.5104 (184.6231)	173.4078 (160.4683)	125.2568 (115.9103)	69.6907 (64.4904)	23.5962 (21.8354)	0.9816 (0.9083)

Applying the parametric spline function approximation in ξ-direction and finite differences (central differences) in θ-direction, Equations (23) and (24) are

$$M_{i,j}\,(1+h\,\beta) = k^2\left[-h\,\alpha\,M_{i+1,j} + \frac{\psi_{i+1,j}-\psi_{i,j}}{h}\right] + \frac{k}{2}\cot\theta_j\,(\psi_{i,j+1}-\psi_{i,j-1}) - (\psi_{i,j+1}-2\psi_{i,j}+\psi_{i,j-1}) + k^2\,\zeta_{i,j}\,e^{3\xi_i}\sin\theta_j, \tag{27}$$

$$\psi_{i,j} = \frac{1}{2}(\psi_{i+1,j}+\psi_{i-1,j}) - \frac{h^2}{2}(\alpha\,M_{i+1,j}+2\beta\,M_{i,j}+\alpha\,M_{i-1,j}), \tag{28}$$

$$k^2\,N_{i,j}\,(1+h\,\beta) = k^2\left[-h\,\alpha\,N_{i+1,j} + \frac{g_{i+1,j}-g_{i,j}}{h}\right] + \frac{k}{2}\cot\theta_j\,(g_{i,j+1}-g_{i,j-1}) - (g_{i,j+1}-2g_{i,j}+g_{i,j-1}), \tag{29}$$

$$g_{i,j} = \frac{1}{2}(g_{i+1,j}+g_{i-1,j}) - \frac{h^2}{2}(\alpha\,M_{i+1,j}+2\beta\,M_{i,j}+\alpha\,M_{i-1,j}), \tag{30}$$

where $i=1,2,\dots,S-1,\ j=1,2,\dots,T-1$.

Two more equations for each of the Equations (25) and (26) can be obtained from the boundary conditions as

$$k^2\,(1+h\,\beta)\,M_{0,j} = k^2\left[-h\,\alpha\,N_{i,j+1} + \frac{\psi_{1,j}}{h}\right] + k^2\,\zeta_{0,j}\sin\theta_j, \tag{31}$$

$$k^2\,(1-h\,\beta)\,M_{S,j} = \frac{k}{2}\cot\theta_j\,(\psi_{S,j+1}-\psi_{S,j-1}) - (\psi_{S,j+1}-2\psi_{S,j}+\psi_{S,j-1}) + k^2\left[h\,\alpha\,M_{S-1,j} + \frac{\psi_{S,j}-\psi_{S-1,j}}{h}\right], \quad \text{and} \tag{32}$$

$$k^2\,(1+h\,\beta)\,N_{0,j} = k^2\left[-h\,\alpha\,N_{1,j} + \frac{g_{1,j}-g_{0,j}}{h}\right] + \frac{k}{2}\cot\theta_j\,(g_{0,j+1}-g_{0,j-1}) - (g_{0,j+1}-2g_{0,j}+g_{0,j-1}), \tag{33}$$

$$k^2\,(1-h\,\beta)\,N_{S-1,j} = k^2\left[h\,\alpha\,N_{S-1,j} - \frac{g_{S-1,j}}{h}\right]. \tag{34}$$

Equations (27) and (28) are solved using the SLOR method with step sizes $h=0.1$ in ξ-direction and $k=\pi/45$ in θ-direction, such that $S\,h=3.0$ and $T\,k=\pi$, taking the starting values for M's and ψ's as zeros. Using these values of ψ's, Equations (29) and (30) are solved. We take the starting values for N's and ζ's as zeros. This completes one iteration. The iterations are continued till $|F^{(n+1)}-F^{(n)}|<10^{-4}$, where $F=M,\ \psi,\ N,\ g$. Table 3 gives values for ψ using the parametric spline. The exact solution is $\psi=\frac{1}{2}\left(r^2-\frac{3}{2}\,r+\frac{1}{2r}\right)\sin^2\theta$. These values are given in brackets. Table 4 gives the values of ψ using the cubic spline. The exact values again are given in brackets. There is a good agreement with the exact solution values by using splines.

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